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CALCULATION OF STEADY TEMPERATURE FIELDS IN GENERALIZED COUETTE FLOWS OF SIMPLE FLUIDS

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Abstract—A basic differential equation of the local balance of the energy flux in homogeneous simple fluids (in Noll's sense) is redeveloped for spatially and materially steady, quasi-simple shearing motions along circular (or also parallel) streamlines. By use of some theorems known from nonlinear continuum theory, it is shown that the supposed motions—here briefly referred to as generalized Couette flows—are dynamically almost possible if they are restricted to very narrow gaps, and that the material response of only first degree in the kinematic tensors, as represented by the viscosity function, covers the phenomena of work flux exhaustively.

The partial differential equation established in a general form is then reduced to an ordinary one by assuming boundary conditions which allow to neglect all derivatives of the temperature other than those with respect to the transverse direction (i.e. perpendicular to the walls of the narrow gap). In the longitudinal direction (i.e. along the streamlines), the temperature does not change owing to the rotational symmetry supposed. In the lateral direction, in which the gap width may slowly change, any edge effects are disregarded. As to the transverse direction, two cases are taken account of: (a) both walls of the gap are maintained at constant, possibly different temperatures; (b) one wall has a fixed temperature, but the other wall is insulated against heat conduction so that its temperature is contingent upon the heat production. Approximate formulas, based on these assumptions and a Taylor expansion in the transverse coordinate, are worked out for temperature fields in gaps of revolution of any cross-sectional shape.

In assigning specific orthogonal coordinate systems fitting to various—namely, cylindric, parallel, conic, spheric, parabolic, hyperbolic, and elliptic—patterns of narrow gaps of revolution, the way is paved for giving definite and mathematically exact solutions. The shear-rate dependence of the viscosity is allowed for in a first approximation based on a power-series expansion in terms of the kinematic invariant. A collection of basic formulas for the diverse patterns of gaps is given mainly for reference purposes. At the close, some often used assemblies (parallel plates, plate and cone) are studied in detail.

NOMENCLATURE

- **A**, tensor of second rank (dimensionally variable);
- c, focal distance [L];
- c_H, specific heat capacity [$\mathrm{L^2\,t^{-2}\,T^{-1}}$];
- C, C_i, constant quantities (dimensionally variable);
- **D**, stretching tensor [$\mathrm{t^{-1}}$];
- f, function defined by (40);
- g, function defined by (42);
- h_i, coordinate scale factor [dimensionally variable];
- **j**, areal density of energy influx [$\mathrm{M\,t^{-3}}$];
- K, coefficient defined by (54);
- L, coefficient defined by (55);
- m, parameter of frame-indifferent derivatives [dimensionless];
- p, hydrostatic pressure [$\mathrm{M\,L^{-1}\,t^{-2}}$];
- r, revolution radius [L];
- s, transverse position variable [dimensionally variable];
- S, gap width [dimensionally variable];
- t, time [t];
- **T**, stress tensor [$\mathrm{M\,L^{-1}\,t^{-2}}$];
- u, internal energy per unit mass [$\mathrm{L^2\,t^{-2}}$];
- v, fluid velocity [$\mathrm{L\,t^{-1}}$];
- **W**, spin tensor [$\mathrm{t^{-1}}$];
- **x**, position vector [L];
- z, axial coordinate [L].

Greek symbols

- α, β, functions defined by (40);
- γ, shear [dimensionless];
- $\Gamma^{(i)}$ material coefficient [$\mathrm{M\,L^{-1}\,t^{n-2}}$; $n = 1,\ldots$];
- ζ, material constant of third degree [$\mathrm{M\,L^{-1}\,t}$];
- η, shear viscosity constant [$\mathrm{M\,L^{-1}\,t^{-1}}$];
- ϑ, elliptic coordinate, also polar distance [dimensionless];
- θ, temperature [T];
- κ, coefficient defined by (53);
- λ, thermal conductivity [$\mathrm{M\,L\,t^{-3}\,T^{-1}}$];
- μ, ν, parabolic coordinates [$\mathrm{L^{1/2}}$];
- ξ, curvilinear orthogonal coordinate [dimensionally variable];
- Ξ, function defined by (79);
- ρ, radial coordinate [L];
- ρ_M, voluminal density of mass [$\mathrm{M\,L^{-3}}$];
- σ, τ, real variables [dimensionless];
- Υ, temperature difference across the gap, divided by the gap width [dimensionally variable];
- X, function defined by (78);
- ψ, hyperbolic coordinate [dimensionless];
- ω, angular velocity [$\mathrm{t^{-1}}$];
- Ω, angular velocity difference across the gap [$\mathrm{t^{-1}}$].

1. INTRODUCTION

WHEN viscous fluids undergo fast shearing, a drop of the apparent viscosity may emerge and entail the question of accounting for this effect either by the assumption of a non-Newtonian flow behavior or by the temperature dependence of the viscosity. By the way, this question has a historical significance in that it was paralleled with the question of the legitimacy of rheology as a branch of science in its own right. By now it has been settled that non-Newtonian phenomena are indeed relevant in a variety of cases, and a scalar nonlinearity of the viscosity will here be taken into account in terms of a workable power-series approximation.

On the other hand, Blok [1], invoking a hypothesis ascribed to Bondi [2], pointed out that viscous heating may prevail at a shearing stress higher than about $5 \times 10^4\,\mathrm{N/m^2}$. This value can only be reached when either the rate of shear is extremely high (such as in high-speed lubrication) or the material has an enormous viscosity (such as in forming processes of chemical engineering).

In the first of these cases the flow mostly occurs between solid walls which are very little apart but extend amply in the other dimensions. Such devices will here be briefly referred to as *narrow gaps*. Yet we come to demonstrate that viscous heating effects are not bound to narrow gaps only, the substantially responsible factors being velocity (not divided by the gap width) and viscosity.

The present paper is primarily to meet the demand for a comprehensive presentation of the relationships governing the temperature distribution in gaps where *simple shearing in a somewhat widened sense* takes place. Our interest is focused on *circular motions* of fluids between rotating solid walls. (The heating phenomena in rectilinear flows through narrow tubes and slits with stationary walls have been analyzed by several workers; e.g. [3] through [5].) A net effect of heat convection is thereby precluded. Hence the thermal process that matters is *heat conduction across the gap* only. The various patterns of gaps of revolution considered in a previous paper [6] are here studied again. Since we assume the fluids to be guided by boundary walls moving against each other at a fixed mutual distance, we suggest for the motions under study the generic term of *generalized Couette flows*. Furthermore, the assumption of a *steady state*—both in spatial and material descriptions—will rule out complications associated with transient phenomena.

The kinematical assumptions—which were already made in the former paper [6]—have, of course, to be checked for being also dynamically possible. This task —though eased by achievements of the previous research of many workers—largely occupies Section 2, while the calculation of definite temperature fields is left to subsequent sections.

Although Section 2 can be read as a self-contained survey of common interest, one of its main purposes shall be to warn the reader against inconsiderately employing the formulas given in Section 5 for gaps whose dimensional proportions are too much in favor of their width. Caution is required mainly because of the tendency to *flow instability*. The interdependence between flow instability and temperature distribution, however, must stay outside the scope of this paper.

In general, the temperature distribution associated with the supposed flow retroacts upon the velocity distribution, and vice versa. This is due mainly to the temperature dependence of the viscosity and, to a less extent, also to thermal conductivity. If there is a pronounced nonlinear flow behavior, the properties of heat transport may undergo a change caused by the flow and thus modify the temperature field, again with the effect of possible retroactions. Such phenomena as well as boundary effects (slip at the gap walls, disturbances at the open edge, and all that) are disregarded here. What comes to be considered represents much as the initial stage of approximation for more sweeping analyses (cf. [7] through [10] for particular cases).

The present investigations may be of particular concern for small gaps, in which accurate temperature measurements are hardly practicable. Instead, calculations can be performed, in some analogy to the shell theory of structural mechanics. It is true, the calculational results can be applied also to large (albeit narrow) gaps, which are amenable to measurements of the temperature distribution, and may stimulate designing what can be called *thermoviscometers*. Such instruments must be sizable so as to accommodate several thermocouples for measuring the temperature distribution across the gap without sensibly disturbing the flow. However, the experimental aspects of thermorheology are not touched upon in this paper. (Regarding the so-called slip problem associated with inhomogeneous flows in wide gaps cf., e.g. [11].)

2. PREREQUISITES

2.1. *Thermodynamical fundamentals*

The principles of the conservation of energy, mass, momentum and angular momentum lead—for a material continuum in which there is no energy absorption by irradiation or from internal energy sources of any kind—to a differential equation of the *local balance of energy flux* (or power), which we put in the somewhat uncommon but compact form

$$\operatorname{div}(\mathbf{j}_W + \mathbf{j}_H) = \rho_M \dot{u} \tag{1}$$

with the denotations $\mathbf{j}_W$ and $\mathbf{j}_H$ for the vectors of the influx densities of work and heat, ρ_M for the mass density, and u for the internal (nonkinetic) energy† per unit mass. The dot put on top indicates the material derivative with respect to the time; for instance,

$$\dot{u} = (\partial u/\partial t) + \dot{\mathbf{x}}^T \operatorname{grad} u. \tag{2}$$

($\dot{\mathbf{x}}$ denotes the velocity vector. The transposition symbol T, intermediate between two vectors, indicates the scalar product of these vectors.)

Assuming *spatially steady processes*, thereby excluding explicit time dependence, means that the partial

† Note that in (1) kinetic energy does not enter at all because its time derivative is cancelled out by part of the mechanical working in virtue of the dynamic balance, Cauchy's first law of motion (cf., e.g. [12], esp. pp. 115–117). Clearly, in contrast to the balance of *energy flux* according to (1), the balance of *energy* does contain the kinetic energy in general. Hence the reader should understand why we desist from the widespread, but improper, usage of designating an equation like (1) as an energy balance.

derivatives with respect to the time, $\partial/\partial t$, disappear. Moreover, the material continuum is supposed to be a *homogeneous body* in regard of its thermomechanical properties. The internal energy of the body may be stored only as heat. Then we can put u proportional to temperature θ, with a proportionality factor C being spatially constant. Hence we get from (2)

$$\dot{u} = C\dot{\mathbf{x}}^T \operatorname{grad}\theta. \tag{3}$$

The assumption of a steady state also implies that the flows be isothermal with respect to the time. Furthermore, all the heat generated in the gap is conducted away through one or either wall of the gap.† The temperature gradient along the streamlines (in the direction of $\dot{\mathbf{x}}$) is, therefore, zero. In other words, the lines of the temperature gradient and the streamlines are mutually orthogonal, viz.

$$\dot{\mathbf{x}}^T \operatorname{grad}\theta = 0. \tag{4}$$

Hence we get from (3)

$$\dot{u} = 0. \tag{5}$$

This amounts to observing no heat convection.

Now we have to specify the l.h.s. of equation (1). In mechanically *nonpolar materials*, in which $\mathbf{j}_W$ is due only to the working of the symmetric stress tensor $\mathbf{T}$, acting upon the velocity gradient tensor grad $\dot{\mathbf{x}}$, we may write

$$\mathbf{j}_W = -\operatorname{grad}(\mathbf{T}\operatorname{grad}\dot{\mathbf{x}}) \tag{6}$$

and hence

$$\operatorname{div}\mathbf{j}_w = -\operatorname{tr}\mathbf{TD} \tag{7}$$

with the stretching tensor $\mathbf{D}$ defined by

$$\mathbf{D} = \tfrac{1}{2}[\operatorname{grad}\dot{\mathbf{x}} + (\operatorname{grad}\dot{\mathbf{x}})^T]. \tag{8}$$

In materials in which $\mathbf{j}_H$ is due only to heat conduction, Fourier's law

$$\mathbf{j}_H = -\boldsymbol{\lambda}\operatorname{grad}\theta \tag{9}$$

holds approximately at small gradients of temperature. In thermally *isotropic materials* the tensor of thermal conductivity, $\boldsymbol{\lambda}$, degenerates to a scalar λ.‡ Under the assumption of material homogeneity already agreed upon, λ is spatially constant. Consequently,

$$\operatorname{div}\mathbf{j}_H = -\lambda\operatorname{div}\operatorname{grad}\theta. \tag{10}$$

Later on, λ is supposed to be also independent of the temperature.

With (5), (7) and (9) we arrive at the differential equation

$$\lambda\operatorname{div}\operatorname{grad}\theta = -\operatorname{tr}\mathbf{TD} \tag{11}$$

(Poisson's equation), which serves to determine a temperature field $\theta(\mathbf{x})$.

2.2. *Kinematical conditions*

In conformity with paper [6], we consider what will be defined here as *quasi-simple shearing* motions, whose streamlines form coaxial circles. (The latter specification is not to preclude rectilinear motions, for the radii of the circles may tend to infinity. But the streamlines must be parallel in this limiting case.) Further as in paper [6], we describe these motions by means of orthogonal curvilinear coordinate systems with lateral coordinates ξ^n (normal to the surfaces of shear), transverse coordinates ξ^t (coinciding with the gradient lines of the shearing motion) and longitudinal coordinates ξ^l (coinciding with the streamlines). The velocity field $\dot{\mathbf{x}}(\mathbf{x})$ is accordingly given by the so-called physical components

$$\dot{x}\langle n\rangle = 0, \quad \dot{x}\langle t\rangle = 0, \quad \dot{x}\langle l\rangle = r\omega \tag{12}$$

with the radial distance r of the circular path from the axis of rotation and with angular velocity ω (which is independent of ξ^l). The only nonzero component of grad $\dot{\mathbf{x}}$, which is called the rate of shear and designated by $\dot{\gamma}$, completely characterizes the simple shearing. We have

$$\dot{x}\langle l,t\rangle = \dot{\gamma} = r\partial\omega/h_t\partial\xi^t \tag{13}$$

since all affinity coefficients of the covariant derivative in orthogonal coordinate systems vanish. (h_m denote the scale factors of orthogonal curvilinear coordinates ξ^m.)

In general, the material derivative $\dot{\mathbf{A}}$ of any tensor $\mathbf{A}$ with respect to the time is defined by

$$\dot{\mathbf{A}} = (\partial\mathbf{A}/\partial t) + (\operatorname{grad}\mathbf{A})\dot{\mathbf{x}} \tag{14}$$

in spatial description. The first term on the r.h.s. of this equation vanishes because of the spatial steadiness already stipulated. For $\mathbf{A}$ we now substitute grad $\dot{\mathbf{x}}$. This tensor has only zero components except for the $\langle lt\rangle$-component given by (13). The spatial gradient component of this tensor along the streamlines—that is, its covariant derivative with respect to ξ^l—vanishes because of the *rotational symmetry* implying

$$\partial/\partial\xi^l = 0. \tag{15}$$

Furthermore, it turns out that all of the affinity coefficients in orthogonal coordinate systems vanish. The result is

$$(\operatorname{grad}\operatorname{grad}\dot{\mathbf{x}})\dot{\mathbf{x}} = \mathbf{0}. \tag{16}$$

† This condition seems to be fairly satisfied in the measuring gaps of thermostated rotatory viscometers or in the lubricating gaps of cooled slide bearings, but is not so well fulfilled between walls made of glass, whose thermal conductivity is about two decimals smaller than that of metals. According to [13] and [14], capillary flows may rather be adiabatic; yet cf. [5].

‡ It should not be withheld that there may be an appreciable anisotropy of thermal properties induced by flow orientation. However, experiments on greases, communicated in [15], have shown that the decrease (9 per cent on the average, after rectifying a misprint) of the thermal conductivity in the directions normal to the streamlines—that is, also in the direction which matters in the problem under study—falls behind the increase (23 per cent on the average) in the longitudinal direction. With $\lambda_\parallel$ denoting the latter, $\lambda_\perp$ denoting the former and λ denoting the thermal conductivity in the isotropic state of rest, those findings obey the theoretical relation

$$(1/\lambda_\parallel) + (2/\lambda_\perp) = 3/\lambda.$$

Hence the material time derivative of grad $\dot{\mathbf{x}}$ vanishes. From (8) we thus obtain

$$\dot{\mathbf{D}} = \mathbf{0}, \qquad \dot{\mathbf{W}} = \mathbf{0} \tag{17}$$

for the stretching tensor **D** and the conjugate antisymmetric tensor, the so-called spin tensor, **W**. The motions characterized by (17) are termed *materially steady* because they exhibit time independence in material description.

(Motions which exhibit time independence in spatial description are customarily designated by the sole word "steady". The motions investigated in the present paper are both spatially and materially steady. Note that both these properties coincide if and only if the velocity gradient component along the streamlines equals zero in spatial description.)

From Euler's criterion for circulation-preserving motions

$$(\operatorname{grad}\ddot{\mathbf{x}})^T = \operatorname{grad}\ddot{\mathbf{x}} \tag{18}$$

follows, in particular, the equivalent criterion $\dot{\mathbf{W}} = \mathbf{0}$ if the tensor of the velocity gradient is nilpotent, viz.

$$(\operatorname{grad}\dot{\mathbf{x}})^2 = \mathbf{0}. \tag{19}$$

Hence we can replace grad $\ddot{\mathbf{x}}$ in [18] by the material time derivative of grad $\dot{\mathbf{x}}$.† The motions under study do satisfy these conditions; that is to say, they are *circulation-preserving*. After a theorem enunciated by Coleman and Truesdell [17], such motions are *dynamically possible* in all homogeneous incompressible simple materials (in Noll's sense) if and only if these motions are also *homogeneous*, viz. in particular, if

$$\operatorname{grad}\operatorname{grad}\dot{\mathbf{x}} = \mathbf{0}. \tag{20}$$

(Note that the homogeneity of *motion* as required by the above equation is by no means tantamount to the *material* homogeneity stipulated in Section 2.1.)

The simplest imaginable, kinematically admissible motions of the type described by (12) and (13) are apparently featured by

$$\partial\omega/\partial\xi^t = \text{const}, \tag{21}$$

while (20) comprises that the rate of shear is constant. These demands can be made approximately consistent only if the factor r/h^t in (13) undergoes but very small variations over the width of the gap filled with the flowing medium. This condition says that but *very narrow gaps* can be admitted.

The requirement of narrowness has also a plausible motivation. In principle, every inhomogeneity occurring in a flow is liable to originate some *instability*. Clearly, the instability will develop less, the narrower the gap is. However, the gap must not become so narrow that the wall boundaries begin to exert an adverse influence of another kind.

Under condition (21) the supposed motions appear as homogeneous only relative to a definite coordinate system; they are, so to speak, but "locally" homogeneous. Accordingly, we are dealing here with an infinitesimal enlargement of the group of simple shearing motions, as defined in [16], beyond the group of homogeneous motions. It is this conceptual extension which we want to express by the epithet "*quasi-simple*".

2.3. *Rheological fundamentals*

The definition of **D** given in (8) yields for odd-numbered exponents ($n_o = 1, 3$, etc.)

$$[\mathbf{D}^{n_o}] = \left(\frac{\dot{\gamma}}{2}\right)^{n_o} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} \tag{22}$$

and for even-numbered exponents ($n_e = 2, 4$, etc.)

$$[\mathbf{D}^{n_e}] = \left(\frac{\dot{\gamma}}{2}\right)^{n_e} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{23}$$

(These and the following matrices hold relative to orthonormal bases which correspond to the indices of the physical components in the sequence $\langle n\rangle, \langle t\rangle, \langle l\rangle$.)

Beside the symmetric kinematic tensors given by (22) and (23), time derivatives of the form

$$\overset{*}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{AW} - \mathbf{WA} + m(\mathbf{AD} + \mathbf{DA}) \tag{24}$$

(cf. [18]) are needed for a frame-indifferent description (i.e. one which is invariant under rigid motions and time shifts). The factor m in (24) denotes an arbitrary scalar parameter, whose value may be dictated by expediency. The arbitrariness exists because m multiplies symmetric tensor binomials composed of frame-indifferent tensors only, so that the essential character of the kinematical description is not encroached upon. The choice $m = 0$ leads to the corotational (Jaumann's) time derivative.

For the motions specified in Section 2.2 we obtain, by inserting (22) and (23) into (24),

$$[\overset{*}{\mathbf{D}}] = \frac{\dot{\gamma}^2}{2} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & m+1 & 0 \\ 0 & 0 & m-1 \end{Vmatrix} \tag{25}$$

as kinematic tensors of second degree,

$$[\overset{**}{\mathbf{D}}] = \frac{\dot{\gamma}^3}{2} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & m^2-1 \\ 0 & m^2-1 & 0 \end{Vmatrix} \tag{26}$$

and

$$[\overset{*}{\mathbf{D}}\mathbf{D} + \mathbf{D}\overset{*}{\mathbf{D}}] = \frac{\dot{\gamma}^3}{2} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & m \\ 0 & m & 0 \end{Vmatrix} \tag{27}$$

as kinematic tensors of third degree,

$$[\overset{***}{\mathbf{D}}] = \frac{\dot{\gamma}^4}{2} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & m^3+m^2-m-1 & 0 \\ 0 & 0 & m^3-m^2-m+1 \end{Vmatrix}, \tag{28}$$

$$[\overset{**}{\mathbf{D}}\mathbf{D} + \mathbf{D}\overset{**}{\mathbf{D}}] = \frac{\dot{\gamma}^4}{2} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & m^2-1 & 0 \\ 0 & 0 & m^2-1 \end{Vmatrix}, \tag{29}$$

† Cf. [16]; but supplement there, on p. 288, an inadvertent omission, namely the statement that the *nilpotency of the velocity gradient tensor*, as expressed by (19), is a necessary condition for the equivalence of the criterion $\dot{\mathbf{W}} = \mathbf{0}$.

$$[\dot{\mathbf{D}}^2] = \frac{\dot{\gamma}^4}{2}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & (m+1)^2 & 0 \\ 0 & 0 & (m-1)^2 \end{Vmatrix} \tag{30}$$

and

$$[\dot{\mathbf{D}}\mathbf{D}^2+\mathbf{D}^2\dot{\mathbf{D}}] = \frac{\dot{\gamma}^4}{2}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & m+1 & 0 \\ 0 & 0 & m-1 \end{Vmatrix} \tag{31}$$

as kinematic tensors of fourth degree. The choice $m = 1$ or -1 makes the time derivatives of higher than first order, viz. $\ddot{\mathbf{D}}$, $\dddot{\mathbf{D}}$ etc., vanish.

We can do without writing down higher-order derivatives of **D** from a reason which can be understood on a general deliberation (cf., e.g. [19], esp. pp. 65–73). The motions under study belong to the class of *motions with constant stretch history*. After Noll and Wang, the response functional of simple materials subjected to such motions in the three-dimensional space is uniquely equivalent to a function of the first three Rivlin–Ericksen tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$. These tensors are defined by $\mathbf{A}_1 = 2\mathbf{D}$, $\mathbf{A}_2 = 2\dot{\mathbf{D}}$, $\mathbf{A}_3 = 2\ddot{\mathbf{D}}$ etc. with $m = 1$. For a special class of steady shearing motions, part of which form the motions specified in Section 2.2, it follows that the kinematic tensors of first and second degrees are quite sufficient.

It is worth mentioning that one can choose, for the minimal representation discussed just now, likewise a set of kinematic tensors with $m = -1$ instead of $+1$. This seems obvious since both the cases rest upon *convective* (Oldroyd's) time derivatives, whose two forms are merely mathematically distinguishable in that the covariant tensor differentiation corresponds to $m = 1$ whereas the contravariant one corresponds to $m = -1$.

On the basis of Wang's corollary, we may thus represent the material response under the specified motions by a function of **D** and $\dot{\mathbf{D}}$ with $m = 1$ or -1. Moreover, supposing *material isotropy*, we invoke a famous theorem of Rivlin and Ericksen, according to which a function whose arguments and values are three-dimensional symmetric tensors can be represented by a tensor polynomial of the second degree in the argument tensors. In the case at hand we obtain for the stress **T** of an incompressible simple fluid under a pressure P

$$\mathbf{T} = -p\mathbf{1}+\Gamma^{(1)}\mathbf{D}+\Gamma^{(11)}\mathbf{D}^2+\Gamma^{(2)}\dot{\mathbf{D}}. \tag{32}$$

The three material coefficients $\Gamma^{(i)}$ constitute scalar-valued functions of the simultaneous invariants of the kinematic tensors.

From (32), with the aid of (22), (23) and (27), it follows that

$$\mathrm{tr}\mathbf{TD} = \Gamma^{(1)}\mathrm{tr}\mathbf{D}^2 \tag{33}$$

with

$$\mathrm{tr}\mathbf{D}^2 = \dot{\gamma}^2/2. \tag{34}$$

It is seen that (33) incorporates no material functions of higher than the first degree. Relying on the generality of the constitutive presuppositions, we may word this result like this: *In steady quasi-simple shearing flows*—which are, by definition, also isochoric, but must be more than locally homogeneous—*the divergence of the work flux density is*, on the part of the material response, *completely determined by the*—in general, nonlinear—*viscosity*, but not by any kind of viscoelasticity whatever.

Besides, the material functions prove to be functions of $\dot{\gamma}^2$ only. By a Taylor expansion about the rest state ($\dot{\gamma} = 0$), which is assumed to be nonsingular, we obtain from (33) in conjunction with (34)

$$\mathrm{tr}\mathbf{TD} = (\eta+\zeta\dot{\gamma}^2)\dot{\gamma}^2 \tag{35}$$

in a first approximation. In case of pseudoplastic fluids, one has $\zeta < 0$. We are allowed to regard η and ζ approximately as material constants, provided $\dot{\gamma}^2 \ll \zeta/\eta$.

3. DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS

On the footing of Section 2 we now advance to work up the differential equation (11), assuming that the material constants do not depend upon the temperature, so that this equation can be dealt with uncoupled from the purely mechanical field equations. By taking advantage of orthogonal curvilinear coordinates ξ^m with scale factors h_m and curvature radii ρ_{pq} (as previously explained in paper [6]) the components of the differential equation under study can be written out as

$$\lambda\sum_p\left(\frac{\partial}{h_p\partial\xi^p}\frac{\partial\theta}{h_p\partial\xi^p}+\sum_q\frac{\partial\theta}{\rho_{pq}h_q\partial\xi^q}\right)+\mathrm{tr}\mathbf{TD} = 0. \tag{36}$$

The sums here have to be formed over all of the three dimensions of space.

In terms of the special coordinate systems (already introduced in Section 2.2) with coordinates ξ^n, ξ^t, ξ^l, the following simplifications can be formulated: first, the rotational symmetry implies that the derivative $\partial/\partial\xi^l$ equals zero in keeping with (15); second, the approximate relation

$$\partial/\partial\xi^n = 0 \tag{37}$$

applies because the supposition of very narrow gaps admits of the assumption that the major values of the temperature gradient occur in the transverse direction. Upon inserting the relations (12) and (21), equation (36) takes on the simple form

$$\lambda\left[\frac{\partial}{h_t\partial\xi^t}\frac{\partial\theta}{h^t\partial\xi^t}+\left(\frac{1}{\rho_{nt}}+\frac{1}{\rho_{lt}}\right)\frac{\partial\theta}{h_t\partial\xi^t}\right]+\mathrm{tr}\mathbf{TD} = 0. \tag{38}$$

It is owing to assumption (37) that the partial differential equation (36) could be converted into the ordinary one (38). Furthermore, that assumption involves a physically significant boundary condition: any influence of the free edge of a gap on the heat transfer appears to be removed out to $\xi^n = \infty$. But what may the lateral heat transfer at the gap edge actually be? Adopting a zero temperature gradient there—as often done for sheer convenience—would presumably be an undervaluation. Also dare we doubt whether it should be possible to realize any fixed temperature there at all. Obviously, the uncertainty of this boundary condition is most reasonably eluded by what we have settled on: the heat transfer near the edge does not appreciably differ from that appearing in the fictitious case of a smooth and unbounded continuation of the gap, together with the medium filling the gap, beyond the real edge. Consequently, *no influence of the edge* is felt.

From (35) and (13) we get

$$\mathrm{tr}\mathbf{TD} = \lambda h_t^{-2} f \tag{39}$$

with

$$f = (\alpha + \beta h_t^{-2} r^2) r^2 \tag{40}$$

and the abbreviations

$$\alpha = (\eta/\lambda)(\partial\omega/\partial\xi^t)^2, \quad \beta = (\zeta/\lambda)(\partial\omega/\partial\xi^t)^4. \tag{41}$$

After the common parlance, f may be called the perturbation function. Since the above specification of this function is but one of many possible ones, we shall not make use of it before there is need of doing so (viz. from Section 4.2 onward).

Symbolizing the derivative with respect to ξ^t by a prime and writing

$$g = h_t\left(\frac{1}{\rho_{nt}} + \frac{1}{\rho_{lt}}\right) - \frac{\partial h_t}{h_t \partial \xi^t}, \tag{42}$$

we obtain from (38) the abbreviated form

$$\theta'' + g\theta' + f = 0. \tag{43}$$

This is a linear second-order differential equation, which will be ready to solve.

The particular solutions looked for are subject to any set of two boundary conditions. With a view to circumstances met in important engineering and measuring appliances, we stipulate—for those cases in which particular solutions will be given here expressly—that in any two surfaces $\xi^t = \xi_0^t$ and $\xi^t = \xi_0^t + S$ (usually those bounding the two gap walls separated by a distance S), for example, the temperature is constant, say

$$\theta(\xi_0^t) = 0, \qquad \theta((\xi_0^t + S)) = \Upsilon S. \tag{44}$$

Here setting the temperature equal to zero at one of the walls means no loss of generality since temperature is physically defined only up to an arbitrary additive constant. An appointment corresponding to (44) will be referred to as the case of *ambidirectional heat transfer*. However, other appointments are legitimate too, and some may be even preferable in certain cases.

We consider a second set of boundary conditions, according to which it suffice—in order to ensure the isothermal state supposed—to let but one of the gap walls have thermal contact with an infinite heat reservoir (as was assumed in the former case for both the walls), while the other wall be kept thermally insulated. Thus the boundary conditions may be exemplified alternatively by

$$\theta(\xi_0^t) = 0, \qquad \theta'((\xi_0^t + S)) = 0. \tag{45}$$

This case of *unidirectional heat transfer* is likely to fit, to some extent, the facts perhaps existing in bearings made of metal-and-nonmetal couples, or in similar devices. We may anticipate that the temperature elevation in the gap will be higher in this case than in the former. (Turian and Bird [8] found that in a rotatory viscometer, composed of a thermostated plate and a thermally insulated cone, the viscous heating effect was even slightly larger than predicted by their theory.)

4. GENERAL SOLUTIONS

4.1. *Solving procedure by a Taylor series approach*

Mathematically exact solutions of the differential equation (43) cannot be given without specifying the functions f and g. But it will become manifest now that a physically meaningful approximation immediately bears out some general results without such specifications, provided the gap width S remains infinitesimal in contrast to the lateral dimension of the gap and provided the quantity Υ stays finite (the latter proviso being intelligible since exceedingly high temperature differences between the gap walls would suppress the observation of heat production effects).

With the transverse positional variable

$$s = \xi^t - \xi_0^t, \tag{46}$$

which remains confined to within the infinitesimal interval

$$0 \leqslant s \leqslant S, \tag{47}$$

we introduce a Taylor expansion of a suitable degree n

$$\theta = \sum_{\nu=1}^{n} \theta_0^{(\nu)} s^\nu/\nu!, \tag{48}$$

assuming $\theta_0 = 0$ in accordance with the first of the boundary conditions (44) or (45) and denoting by $\theta_0^{(\nu)}$ the value of the νth derivative of θ with respect to s at $s = 0$.

(a) *Case of ambidirectional heat transfer.* On account of the second of the boundary conditions (44), which prescribes $\theta = \Upsilon S$ at $s = S$, we have from (48) as a first equation for the determination of the constant coefficients $\theta_0^{(\nu)}$

$$\theta_0' + \tfrac{1}{2}\theta_0'' S + \tfrac{1}{6}\theta_0''' S^2 = \Upsilon, \tag{49}$$

when expanding only up to the third degree in s (including S). The differential equation (43) merely serves to deliver the rest of the conditional equations for the coefficients $\theta_0^{(\nu)}$. Let f_0, g_0 denote the values of the functions f, g, and let f_0', g_0' denote the values of the derivatives of f, g with respect to s, taken at $s = 0$. Then we immediately obtain from (43)

$$g_0\theta_0' + \theta_0'' = -f_0 \tag{50}$$

and further, by differentiating (43) and evaluating at $s = 0$,

$$g_0'\theta_0' + g_0\theta_0'' + \theta_0''' = -f_0'. \tag{51}$$

Resolving the system of conditional equations (49) through (51), dropping all terms of degree higher than 3 (while assigning to Υ the degree 0, at the least) and substituting the expressions thus obtained for the coefficients $\theta_0^{(\nu)}$ into (48), we arrive at the approximate solution

$$\theta = \Upsilon S + \left[\frac{\kappa}{2}K + \tfrac{1}{6}L(S+s)\right](S-s)s \tag{52}$$

with the abbreviations

$$\kappa = 1 + \tfrac{1}{2}g_0 S, \tag{53}$$

$$K = f_0 + g_0\Upsilon \tag{54}$$

and

$$L = f_0' - f_0 g_0 + (g_0' - g_0^2)\Upsilon. \tag{55}$$

(This result can likewise be gained by the perhaps more habitual procedure of setting out with general Taylor expansions for f and g, as well as for θ, in the differential equation and making a successive comparison of coefficients.)

(b) *Case of unidirectional heat transfer.* In place of (49) we find on account of (45)

$$\theta_0' + \theta_0'' S + \tfrac{1}{2}\theta_0''' S^2 = 0, \tag{56}$$

whereas (14) and (15) continue to hold. In analogy to case (a), the approximate solution comes out to be

$$\theta = f_0(S - \tfrac{1}{2}s)s + \tfrac{1}{2}[f_0'(S^2 - \tfrac{1}{3}s^2) + f_0 g_0(S^2 - Ss + \tfrac{1}{3}s^2)]s. \tag{57}$$

4.2. *Characteristics of the approximate temperature profiles*

(a) *Case of ambidirectional heat transfer.* The accuracy of the approximate solutions worked out above is more restricted, the higher the order of the derivative of the quantity under investigation. This fact must be borne in mind when the characteristics of the temperature profile across a gap come to be ascertained in terms of the derivatives, which read in the present case on the basis of the solution (52)

$$\theta' = \Upsilon + \frac{\kappa}{2} K(S - 2s) + \tfrac{1}{6}L(S^2 - 3s^2) \tag{58}$$

and

$$\theta'' = -\kappa K - Ls. \tag{59}$$

Setting $\theta' = 0$ and supposing $L \neq 0$, we find the solution of the *third-degree approximation* according to (58) to be

$$s_{\theta\,\max} = -\frac{\kappa K}{L} + \left[\left(\frac{\kappa K}{L}\right)^2 + \frac{2\Upsilon}{L} + \frac{\kappa K}{L}S + \tfrac{1}{3}S^2\right]^{1/2} \tag{60}$$

for the location $s = s_{\theta\,\max}$ of a possible temperature maximum, which actually exists if the value $s_{\theta\,\max}$ proves to lie inside the interval (47). (The proper sign of the square root in the above equation can be checked in a way exemplified in Section 6.2.) The particular case $L = 0$ yields the limiting value

$$s_{\theta\,\max} = \frac{S}{2} + \frac{\Upsilon}{\kappa K}. \tag{61}$$

Setting $\theta'' = 0$, we find for possibly existing points of inflection in the temperature profile

$$s_{\theta\,\mathrm{infl}} = -\kappa K/L. \tag{62}$$

When only the *second-degree approximation*, viz.

$$\theta = [\Upsilon + \tfrac{1}{2}K(S - s)]s, \tag{63}$$

is drawn upon, no points of inflection come to the fore. The location of a possibly existing temperature maximum is then given by

$$s_{\theta\,\max} = (S/2) + (\Upsilon/K), \tag{64}$$

which differs from (61) merely in that the number 1 occupies the place of κ. At this approximation stage the maximum is easily calculated to be

$$\theta_{\max} = K(s_{\theta\,\max})^2/2 \tag{65}$$

with $s_{\theta\,\max}$ substituted from (64). These simple formulas readily lend themselves to discovering some essential features, which will now be set forth.

To begin with, we have to specify K according to (54). When Υ tends to zero, the influence of the purely geometrical quantity g_0 vanishes, so that f_0 alone retains importance. We will contemplate this case at first.

Assuming $\zeta = 0$ (thereby excluding nonlinear viscosity), we obtain from (40) in conjunction with (41)

$$f = (\eta/\lambda)(r\partial\omega/\partial\xi')^2. \tag{66}$$

Furthermore, let $\partial\omega/\partial\xi'$ be independent of ξ' (i.e. the velocity profile be linear), say, equal to Ω/S with Ω denoting the difference of the angular velocities of both the gap walls. Consequently,

$$f = (\eta/\lambda)(r\Omega/S)^2. \tag{67}$$

If $\Upsilon = 0$, we get from (54) $K = f_0$, and thus from (65) in conjunction with (64) and (67)

$$\theta_{\max} = (\eta/8\lambda)(r_0\Omega)^2. \tag{68}$$

This formula has to be looked upon as representative of an outcome emerging from the thermal balance between viscous heat production and conductive heat transfer. Above all, note that the gap width S does not figure in this formula any more. It is true that a large velocity change over a short distance (in other words, a high rate of shear) must occur so as to produce a high viscous heating. But the cooling in the heat-conducting medium, embedded by heat-absorbing walls, goes up equally when the gap width is diminished. As a result, *the temperature rise fails to depend upon the gap width.*

To afford a numerical estimation, one must take into account that the viscosity constant η encompasses an incomparably wide range of values. The thermal conductivity λ, on the contrary, ranges between 0·1 and 0·2 W/mK for most of the liquids, except for water and alcohols, and decreases not more than about 10^{-3}/K (whereas water exhibits an increase) with rising temperature.† For instance, the values $\eta = 1\,\mathrm{Ns/m^2}$ (= 10 P), $\lambda = 0{\cdot}14$ W/mK (as found for lubricating oils within a very wide range of viscosity values) and $r\Omega = 1$ m/s yield the result $\theta_{\max} \approx 1$ K. There may occur higher temperature elevations in more viscous materials or at higher velocity gradients. In gas-filled gaps, however, the much lower viscosity of the gases will commonly not admit of appreciable temperature elevations—in consideration of the facts that most gases, including air, have thermal conductivities of about a tenth, and that hydrogen has one of like order of magnitude, compared

† All known theories of heat conduction intelligibly adopt proportionality of the thermal conductivity to the sound speed (as being identical to the speed of heat waves). Moreover, after a theory which was published by Bridgman as early as in 1923, but seems still approved despite its naive foundation, the thermal conductivity varies inversely as the square of the mean distance between the centers of adjacent molecules. The above-mentioned relations can be roughly deduced from this theory (cf., e.g. [20], esp. pp. 77–83). In contrast, we are unaware of any similar theory concerning viscosity.

to the value given above. (A well-known exception, which results from the huge velocities involved, is encountered at the re-entry of astronautical objects falling into the atmosphere of the earth.)

It is instructive to contrast the effects of heat conduction and convection with each other. The conductive heat flux is defined by (9). The convective heat flux density equals $\rho_M c_H \theta_{abs} v$, with mass density ρ_M, specific heat c_H, absolute temperature θ_{abs} and convective velocity v. Both the heat fluxes become equal to each other in magnitude if $|v|$ takes on the critical value

$$v_{crit} = (\lambda/\rho_M c_H \theta_{abs})|\text{grad}\,\theta|. \quad (69)$$

Let v be a transverse perturbation component of the circular motion of the fluid. For a rough estimation we suppose that a constant temperature gradient exists in the gap between a wall and the locus of the temperature maximum. Assuming $\theta_{abs} = 300\,K$ and a temperature rise of 1 K at a distance of 10^{-4} m, we obtain for a fluid (e.g. a lubricating oil) with $\lambda = 0{\cdot}14$ W/mK, $\rho_M = 920\,\text{kg/m}^3$ and $c_H = 1700$ Ws/kgK the result $v_{crit} = 3 \times 10^{-4}$ m/s. This value looks small as compared to a longitudinal velocity difference which lies in the order of magnitude of 1 m/s (cf. the preceding numerical example). When the gap width increases, v_{crit} decreases inversely proportional to it. Hence we conclude that *even a slight deviation from the uniform motion* due to some flow instability *is able to upset the temperature profile* predicted by the theory, unless the gap is sufficiently narrow.

Having discussed some relevant phenomena on the basis of the simplest case, we are free to pass over to the case $\Upsilon \neq 0$. Clearly, setting up a *temperature difference between the gap walls* effectuates a displacement of the temperature maximum in a direction depending on the sign of the difference. At certain amounts of this difference the maximum disappears beyond one or the other of the walls. There are two critical values of Υ for these events: $\Upsilon = \Upsilon_0$ for $s_{\theta\,max} = 0$ and $\Upsilon = \Upsilon_S$ for $s_{\theta\,max} = S$. In the second-degree approximation we easily calculate from (64) in conjunction with (54) that

$$\Upsilon_0 = -\frac{f_0 S}{2 - g_0 S}, \qquad \Upsilon_S = \frac{f_0 S}{2 + g_0 S}. \quad (70)$$

The criterion for the exclusion of a temperature maximum in the gap may be summarized in terms of the inequality $\Upsilon_0 \leqslant \Upsilon \leqslant \Upsilon_S$. A similar reasoning applies to the existence of possible points of inflection, whose location may be calculated with the aid of (62) at the stage of the third-degree approximation.

(b) *Case of unidirectional heat transfer.* This case can be handled more briefly, for it is in close analogy to the former case, so that a repetition of the commentary may be largely dispensed with. Additionally, major simplifications ensue from the unidirectionality of the heat transfer.

We have merely to state that from (57) and the derivatives

$$\theta' = f_0(S-s) + \tfrac{1}{2}[f_0'(S^2 - s^2) + f_0 g_0 (S-s)^2] \quad (71)$$

and

$$\theta'' = -f_0 - f_0' s - f_0 g_0 (S-s) \quad (72)$$

it follows that

$$s_{\theta\,max} = S \quad (73)$$

and, in the third-degree approximation,

$$s_{\theta\,max} = \frac{2f_0 + (f_0 g_0 + f_0')S}{f_0 g_0 - f_0'} \quad (74)$$

for another temperature maximum and

$$s_{\theta\,infl} = \frac{f_0 + f_0 g_0 S}{f_0 g_0 - f_0'} \quad (75)$$

for points of inflection, possibly appearing within the gap. The temperature maximum at the insulating wall exists under any circumstances and has the value given by

$$\theta_{max} = \tfrac{1}{2} f_0 S^2 + \tfrac{1}{6}(f_0 g_0 + 2f_0')S^3. \quad (76)$$

Note that the leading term of this expression differs from formula (65) only by a constant factor.

5. SPECIAL SOLUTIONS

5.1. *Preliminaries*

The solutions presented in Section 4 are general in that they hold in any orthogonal coordinate system. This generality has to be paid for with a loss in accuracy. However, the underlying differential equation can be solved exactly in particular cases, and these solutions lend themselves to checking the accuracy attained by the approximate formulas of Section 4.

In the sequel is given a compilation of examples, which fully correspond with the patterns considered in paper [6], where the reader can also look up illustrations of the various patterns and basic relations of the calculus operating in the various orthogonal curvilinear coordinate systems.

The selection of the examples was dictated by the intention to investigate devices of some practical importance, namely such as are applicable or factually applied in viscometry, lubrication, or chemical engineering. Patterns which seem difficult to materialize (e.g. toroidal gaps) are not taken into consideration.

Every example of the collection (Section 5.2) will contain, first, the expressions for the functions f and g as defined by (40) and (42), which are needed also for the approximate calculations according to Section 4 in definite instances. Next, the mathematically exact solutions of differential equation (43) will be quoted. For the sake of brevity we forbear displaying the respective solving methods (which may follow standard rules). From a similar reason, only the general integrals are given. Their integration constants, C_1 and C_2, contingent upon particular boundary conditions, are left to be determined for any instance of interest.

It will certainly not be gratuitous to emphasize that the mathematically exact solutions are liable to lose their physical significance when they come to be applied onto gaps of finite width. This statement is a substantial conclusion to be drawn from Section 2.

Unless nonlinear viscosity is excluded, one may be faced—in some instances—with the necessity of computing integrals which defy being expressed solely in terms of elementary functions. These are the natural dilogarithm†

$$\operatorname{diln}\tau = -\int_{-\infty}^{\ln\tau} \ln(1-\tau).\,\mathrm{d}\ln\tau \tag{77}$$

and the integrals

$$\begin{aligned} \mathrm{X}(\tau;\sigma) &= \int_0^{\arctan\tau} \arctan\frac{\tau}{\sigma}.\,\mathrm{d}\arctan\tau \\ &= -\int_0^{\operatorname{artanh}\tau} \operatorname{artanh}\frac{\tau}{\sigma}.\,\mathrm{d}\operatorname{artanh}\tau, \end{aligned}\tag{78}$$

$$\begin{aligned} \Xi(\tau;\sigma) &= \int_0^{\arctan\tau} \operatorname{artanh}\frac{\tau}{\sigma}.\,\mathrm{d}\arctan\tau \\ &= -\int_0^{\operatorname{artanh}\tau} \arctan\frac{\tau}{\sigma}.\,\mathrm{d}\operatorname{artanh}\tau \end{aligned}\tag{79}$$

(with τ and σ denoting any dimensionless real variables). If the integration ranges over but an infinitesimal interval from $\tau = \tau_0$ to $\tau = \tau_0 + \delta\tau$, the integrand can be expanded in a Taylor series in terms of powers of $\delta\tau$ and then be integrated over $\delta\tau$. Expanding up to the second degree, we obtain the general approximation formula

$$\begin{aligned} &\int_{\chi(\tau_0)}^{\chi((\tau_0+\delta\tau))} \varphi(\tau)\,\mathrm{d}\chi(\tau) \\ &= \int_{\tau_0}^{\tau_0+\delta\tau} \varphi(\tau)\chi'(\tau)\,\mathrm{d}\tau \approx \varphi(\tau_0)\chi'(\tau_0)\delta\tau \\ &\quad + \frac{1}{2}[\varphi'(\tau_0)\chi'(\tau_0) + \varphi(\tau_0)\chi''(\tau_0)](\delta\tau)^2 + \frac{1}{3!} \\ &\quad \times [\varphi''(\tau_0)\chi'(\tau_0) + 2\varphi'(\tau_0)\chi''(\tau_0) + \varphi(\tau_0)\chi'''(\tau_0)](\delta\tau)^3, \end{aligned}\tag{80}$$

which may help to avoid laborious evaluations of those transcendental integrals.‡

5.2. *Collection of mathematically exact formulas*

5.2.1. *Cylindric gap.* The walls are bounded by the curved surfaces of coaxial circular cylinders. The surfaces of shear form parallel planes. See Fig. 1.

$$\xi^t = r, \qquad \xi^n = z \tag{81}$$

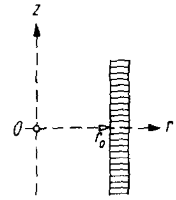

FIG. 1. Cylindric gap.

$$h_t = 1 \tag{82}$$

$$\rho_{nt} = \infty \tag{83}$$

$$\rho_{lt} = r \tag{84}$$

$$f = (\alpha + \beta r^2)r^2 \tag{85}$$

$$g = 1/r \tag{86}$$

$$\frac{\partial\theta}{\partial r} = \frac{C_1}{r} - \frac{\alpha}{4}r^3 - \frac{\beta}{6}r^5 \tag{87}$$

$$\theta = C_2 + C_1 \ln r - \frac{\alpha}{16}r^4 - \frac{\beta}{36}. \tag{88}$$

5.2.2. *Parallel gap.* The walls are bounded by parallel planes. The surfaces of shear constitute coaxial circular cylinders. See Fig. 2.

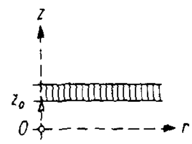

FIG. 2. Parallel gap.

$$\xi^t = z, \qquad \xi^n = r \tag{89}$$

$$h_t = 1 \tag{90}$$

$$\rho_{nt} = \infty \tag{91}$$

$$\rho_{lt} = \infty \tag{92}$$

$$f = (\alpha + \beta r^2)r^2 \tag{93}$$

$$g = 0 \tag{94}$$

$$\frac{\partial\theta}{\partial z} = C_1 - (\alpha r^2 + \beta r^4)z \tag{95}$$

$$\theta = C_2 + C_1 z - \tfrac{1}{2}(\alpha r^2 + \beta r^4)z^2 \tag{96}$$

† The notation of this function is not uniform in the literature (cf. a recent monograph [21]). The function as defined by (77)—in agreement with Nielsen [22], among others—has been tabulated by several workers [23] through [25], but not beyond the domain $-5 \leqslant \tau \leqslant 1$.

‡ This may be illustrated by way of an example. In so doing, we use the parabolic gap formula (120) subsequently listed in Subsection 5.2.5. One has to compute the difference of $\operatorname{diln}(-\mu^{-2}\nu^2)$, as defined by (77), between the boundary points $\nu = \nu_0$ and the intermediate gap points $\nu = \nu_0 + s$ for $\mu = \text{const}$, say, $\mu = \mu_0$ (which represents that surface of shear passing through both the plane $z = 0$ and the bounding surface $\nu = \nu_0$), so that $\tau_0 = -1$. Equation (80), after being rearranged according to Horner's scheme (destined to abridge the computational procedure), then reads

$$\begin{aligned} &\operatorname{diln}\left[-\left(1+\frac{s}{\nu_0}\right)^2\right] - \operatorname{diln}(-1) \\ &\quad \approx [0{\cdot}6931472 + (0{\cdot}0965736 + 0{\cdot}0227157\delta\tau)\delta\tau]\delta\tau \end{aligned}$$

with

$$\delta\tau = -\left(2 + \frac{s}{\nu_0}\right)\frac{s}{\nu_0}.$$

Substituting, for instance, $s = \nu_0/10$ and therefore

$$\delta\tau = -0{\cdot}21,$$

we obtain

$$\operatorname{diln}(-1{\cdot}21) - \operatorname{diln}(-1) \approx -0{\cdot}1415124.$$

This approximate value can be checked with the aid of Powell's table [23], from which we take the values $\operatorname{diln}(-1{\cdot}21) = -0{\cdot}9639673$ (after a quadratic interpolation) and $\operatorname{diln}(-1) = -0{\cdot}8224670$ (by direct reading), hence the sought difference $-0{\cdot}1415003$. Thus we may expect the above approximation to be accurate up to the fourth decimal place. One of the merits of this approximative computation consists in sparing possible entanglements with small differences of large numbers.

5.2.3. *Conic gap.* The walls are bounded by the curved surfaces of convertical and coaxial circular cones. The surfaces of shear constitute concentric spheres. See Fig. 3.

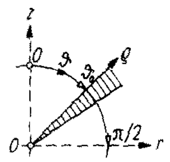

FIG. 3. Conic gap.

$$\xi^t = \vartheta, \qquad \xi^n = \rho, \qquad r = \rho \sin\vartheta \tag{97}$$

$$h_t = \rho \tag{98}$$

$$\rho_{nt} = \infty \tag{99}$$

$$\rho_{lt} = \rho \tan\vartheta \tag{100}$$

$$f = [\alpha + \beta(\sin\vartheta)^2](\rho \sin\vartheta)^2 \tag{101}$$

$$g = \cot\vartheta \tag{102}$$

$$\frac{\partial\theta}{\partial\vartheta} = -\frac{C_1}{\sin\vartheta} + \rho^2\left[\left(\frac{\alpha}{3}+\frac{7\beta}{15}\right)(\cos\vartheta\,.\,\sin\vartheta) - \frac{\beta}{5}(\cos\vartheta)^3\sin\vartheta + \left(\frac{2\alpha}{3}+\frac{8\beta}{15}\right)\cot\vartheta\right] \tag{103}$$

$$\theta = C_2 + C_1 \operatorname{artanh}\cos\vartheta - \rho^2\left[\left(\frac{\alpha}{6}+\frac{7\beta}{30}\right)(\cos\vartheta)^2 - \frac{\beta}{20}(\cos\vartheta)^4 - \left(\frac{2\alpha}{3}+\frac{8\beta}{15}\right)\ln\sin\vartheta\right] \tag{104}$$

5.2.4. *Spheric gap.* The walls are bounded by surfaces of concentric spheres. The surfaces of shear constitute convertical and coaxial circular cones. See Fig. 4.

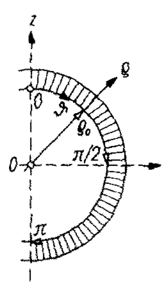

FIG. 4. Spheric gap.

$$\xi^t = \rho, \qquad \xi^n = \vartheta, \qquad r = \rho\sin\vartheta \tag{105}$$

$$h_t = 1 \tag{106}$$

$$\rho_{nt} = \rho \tag{107}$$

$$\rho_{lt} = \rho \tag{108}$$

$$f = [\alpha + \beta(\rho\sin\vartheta)^2](\rho\sin\vartheta)^2 \tag{109}$$

$$g = 2/\rho \tag{110}$$

$$\frac{\partial\theta}{\partial\rho} = \frac{C_1}{\rho^2} - \frac{\alpha}{5}(\sin\vartheta)^2\rho^3 - \frac{\beta}{7}(\sin\vartheta)^4\rho^5 \tag{111}$$

$$\theta = C_2 - \frac{C_1}{\rho} - \frac{\alpha}{20}(\sin\vartheta)^2\rho^4 - \frac{\beta}{42}(\sin\vartheta)^4\rho^6. \tag{112}$$

5.2.5. *Parabolic gap.* Both the bounding surfaces of the walls and the surfaces of shear constitute confocal and coaxial circular paraboloids. See Fig. 5.

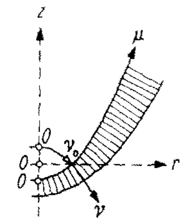

FIG. 5. Parabolic gap.

$$\xi^t = \nu, \qquad \xi^n = \mu, \qquad r = \mu\nu \tag{113}$$

$$h_t = (\mu^2+\nu^2)^{1/2} \tag{114}$$

$$\rho_{nt} = \nu^{-1}(\mu^2+\nu^2)^{3/2} \tag{115}$$

$$\rho_{lt} = \nu(\mu^2+\nu^2)^{1/2} \tag{116}$$

$$f = \left(\alpha + \beta\frac{\mu^2\nu^2}{\mu^2+\nu^2}\right)\mu^2\nu^2 \tag{117}$$

$$g = 1/\nu \tag{118}$$

$$\frac{\partial\theta}{\partial\nu} = \frac{C_1}{\nu} + \frac{\beta}{2}\mu^6\nu - \frac{1}{4}(\alpha+\beta\mu^2)\mu^2\nu^3 - \frac{\beta}{2}\mu^8\nu^{-1}\ln(1+\mu^{-2}\nu^2) \tag{119}$$

$$\theta = C_2 + C_1\ln\nu + \frac{\beta}{4}\mu^6\nu^2 - \frac{1}{16}(\alpha+\beta\mu^2)\mu^2\nu^4 + \frac{\beta}{4}\mu^8 \operatorname{diln}(-\mu^{-2}\nu^2) \tag{120}$$

5.2.6. *Hyperbolic gap of the first kind.* The walls are bounded by surfaces of confocal circular hyperboloids of one sheet. The surfaces of shear constitute confocal oblate spheroids. See Fig. 6.

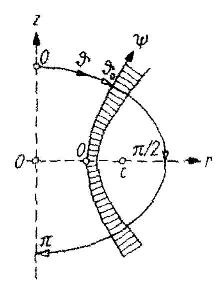

FIG. 6. Hyperbolic gap of the first kind.

$$\xi^t = \vartheta, \qquad \xi^n = \psi, \qquad r = c\cosh\psi\,.\,\sin\vartheta \tag{121}$$

$$h_t = c[(\sinh\psi)^2 + (\cos\vartheta)^2]^{1/2} \tag{122}$$

$$\rho_{nt} = -c(\sin\vartheta\,.\,\cos\vartheta)^{-1}[(\sinh\psi)^2+(\cos\vartheta)^2]^{3/2} \tag{123}$$

$$\rho_{lt} = c\tan\vartheta\,.\,[(\sinh\psi)^2+(\cos\vartheta)^2]^{1/2} \tag{124}$$

$$f = \left[\alpha + \beta\frac{(\cosh\psi\,.\,\sin\vartheta)^2}{(\sinh\psi)^2+(\cos\vartheta)^2}\right](c\cosh\psi\,.\,\sin\vartheta)^2 \tag{125}$$

$$g = \cot\vartheta \tag{126}$$

$$\frac{\partial\theta}{\partial\vartheta} = -\frac{C_1}{\sin\vartheta} + c^2\left\{\frac{1}{3}\left[\alpha(\cosh\psi)^2 - \beta(\cosh\psi)^4\right](\cos\vartheta\,.\,\sin\vartheta) + \frac{1}{3}\left[2\alpha(\cosh\psi)^2 - 2\beta(\cosh\psi)^4 - 3\beta(\cosh\psi)^6\right] \times \cot\vartheta + \frac{\beta}{\sinh\psi}(\cosh\psi)^8(\sin\vartheta)^{-1}\arctan\frac{\cos\vartheta}{\sinh\psi}\right\} \tag{127}$$

$$\theta = C_2 + C_1\,\mathrm{artanh}\cos\vartheta - c^2\left\{\frac{1}{6}\left[\alpha(\cosh\psi)^2 - \beta(\cosh\psi)^4\right](\cos\vartheta)^2 - \frac{1}{3}\left[2\alpha(\cosh\psi)^2 - 2\beta(\cosh\psi)^4 - 3\beta(\cosh\psi)^6\right] \times \ln\sin\vartheta - \frac{\beta}{\sinh\psi}(\cosh\psi)^8\Xi(\cos\vartheta;\sinh\psi)\right\} \tag{128}$$

5.2.7. *Hyperbolic gap of the second kind.* The walls are bounded by surfaces of confocal and coaxial circular hyperboloids of two sheets. The surfaces of shear constitute confocal prolate spheroids. See Fig. 7.

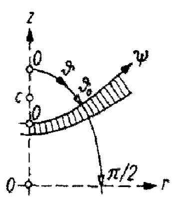

FIG. 7. Hyperbolic gap of the second kind.

$$\xi' = \vartheta, \qquad \xi'' = \psi, \qquad r = c\sinh\psi\,.\,\sin\vartheta \tag{129}$$

$$h_t = c\left[(\sinh\psi)^2 + (\sin\vartheta)^2\right]^{1/2} \tag{130}$$

$$\rho_{nt} = c(\sin\vartheta\,.\,\cos\vartheta)^{-1}\left[(\sinh\psi)^2 + (\sin\vartheta)^2\right]^{3/2} \tag{131}$$

$$\rho_{lt} = c\tan\vartheta\,.\left[(\sinh\psi)^2 + (\sin\vartheta)^2\right]^{1/2} \tag{132}$$

$$f = \left[\alpha + \beta\frac{(\sinh\psi\,.\,\sin\vartheta)^2}{(\sinh\psi)^2 + (\sin\vartheta)^2}\right](c\sinh\psi\,.\,\sin\vartheta)^2 \tag{133}$$

$$g = \cot\vartheta \tag{134}$$

$$\frac{\partial\theta}{\partial\vartheta} = -\frac{C_1}{\sin\vartheta} + c^2\left\{\frac{1}{3}\left[\alpha(\sinh\psi)^2 + \beta(\sinh\psi)^4\right](\cos\vartheta\,.\,\sin\vartheta) + \frac{1}{3}\left[2\alpha(\sinh\psi)^2 + 2\beta(\sinh\psi)^4 - 3\beta(\sinh\psi)^6\right] \times \cot\vartheta + \frac{\beta}{\cosh\psi}(\sinh\psi)^8(\sin\vartheta)^{-1}\,\mathrm{artanh}\frac{\cos\vartheta}{\cosh\psi}\right\} \tag{135}$$

$$\theta = C_2 + C_1\,\mathrm{artanh}\cos\vartheta - c^2\left\{\frac{1}{6}\left[\alpha(\sinh\psi)^2 + \beta(\sinh\psi)^4\right](\cos\vartheta)^2 - \frac{1}{3}\left[2\alpha(\sinh\psi)^2 + 2\beta(\sinh\psi)^4 - 3\beta(\sinh\psi)^6\right] \times \ln\sin\vartheta - \frac{\beta}{\cosh\psi}(\sinh\psi)^8\mathrm{X}(\cos\vartheta;\cosh\psi)\right\} \tag{136}$$

5.2.8. *Elliptic gap of the first kind.* The walls are bounded by surfaces of confocal oblate spheroids. The surfaces of shear constitute confocal circular hyperboloids of one sheet. See Fig. 8.

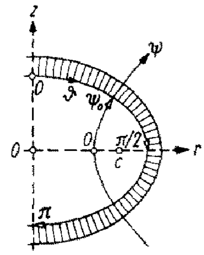

FIG. 8. Elliptic gap of the first kind.

$$\xi' = \psi, \qquad \xi'' = \vartheta, \qquad r = c\sin\vartheta\,.\cosh\psi \tag{137}$$

$$h_t = c\left[(\cos\vartheta)^2 + (\sinh\psi)^2\right]^{1/2} \tag{138}$$

$$\rho_{nt} = c(\sinh\psi\,.\cosh\psi)^{-1}\left[(\cos\vartheta)^2 + (\sinh\psi)^2\right]^{3/2} \tag{139}$$

$$\rho_{lt} = c\coth\psi\,.\left[(\cos\vartheta)^2 + (\sinh\psi)^2\right]^{1/2} \tag{140}$$

$$f = \left[\alpha + \beta\frac{(\sin\vartheta\,.\cosh\psi)^2}{(\cos\vartheta)^2 + (\sinh\psi)^2}\right](c\sin\vartheta\,.\cosh\psi)^2 \tag{141}$$

$$g = \tanh\psi \tag{142}$$

$$\frac{\partial\theta}{\partial\psi} = \frac{C_1}{\cosh\psi} - c^2\left\{\frac{1}{3}\left[\alpha(\sin\vartheta)^2 + \beta(\sin\vartheta)^4\right](\sinh\psi\,.\cosh\psi) + \frac{1}{3}\left[2\alpha(\sin\vartheta)^2 + 2\beta(\sin\vartheta)^4 + 3\beta(\sin\vartheta)^6\right]\tanh\psi + \frac{\beta}{\cos\vartheta}(\sin\vartheta)^8(\cosh\psi)^{-1}\arctan\frac{\sinh\psi}{\cos\vartheta}\right\} \tag{143}$$

$$\theta = C_2 + C_1\arctan\sinh\psi - c^2\left\{\frac{1}{6}\left[\alpha(\sin\vartheta)^2 + \beta(\sin\vartheta)^4\right](\sinh\psi)^2 + \frac{1}{3}\left[2\alpha(\sin\vartheta)^2 + 2\beta(\sin\vartheta)^4 + 3\beta(\sin\vartheta)^6\right]\ln\cosh\psi + \frac{\beta}{\cos\vartheta}(\sin\vartheta)^8\mathrm{X}(\sinh\psi;\cos\vartheta)\right\} \tag{144}$$

5.2.9. *Elliptic gap of the second kind.* The walls are bounded by surfaces of confocal prolate spheroids. The surfaces of shear constitute confocal and coaxial circular hyperboloids of two sheets. See Fig. 9.

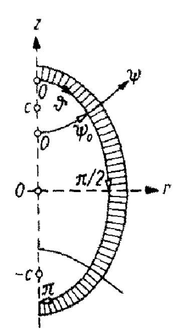

FIG. 9. Elliptic gap of the second kind.

$$\xi^t = \psi, \qquad \xi^n = \vartheta, \qquad r = c\sin\vartheta\,.\sinh\psi \quad (145)$$

$$h_t = c[(\sin\vartheta)^2 + (\sinh\psi)^2]^{1/2} \quad (146)$$

$$\rho_{nt} = c(\sinh\psi\,.\cosh\psi)^{-1}[(\sin\vartheta)^2 + (\sinh\psi)^2]^{3/2} \quad (147)$$

$$\rho_{lt} = c\tanh\psi\,.[(\sin\vartheta)^2 + (\sinh\psi)^2]^{1/2} \quad (148)$$

$$f = \left[\alpha + \beta\frac{(\sin\vartheta\,.\sinh\psi)^2}{(\sin\vartheta)^2 + (\sinh\psi)^2}\right](c\sin\vartheta\,.\sinh\psi)^2 \quad (149)$$

$$g = \coth\psi \quad (150)$$

$$\frac{\partial\theta}{\partial\psi} = -\frac{C_1}{\sinh\psi} - c^2\left\{\frac{1}{3}[\alpha(\sin\vartheta)^2 + \beta(\sin\vartheta)^4](\cosh\psi\,.\sinh\psi) - \frac{1}{3}[2\alpha(\sin\vartheta)^2 + 2\beta(\sin\vartheta)^4 + 3\beta(\sin\vartheta)^6]\coth\psi - \frac{\beta}{\cos\vartheta}(\sin\vartheta)^8(\sinh\psi)^{-1}\operatorname{artanh}\frac{\cosh\psi}{\cos\vartheta}\right\} \quad (151)$$

$$\theta = C_2 + C_1\operatorname{artanh}\cosh\psi - c^2\left\{\frac{1}{6}[\alpha(\sin\vartheta)^2 + \beta(\sin\vartheta)^4](\cosh\psi)^2 - \frac{1}{3}[2\alpha(\sin\vartheta)^2 + 2\beta(\sin\vartheta)^4 + 3\beta(\sin\vartheta)^6]\ln\sinh\psi - \frac{\beta}{\cos\vartheta}(\sin\vartheta)^8 X(\cosh\psi;\cos\vartheta)\right\}. \quad (152)$$

5.3. *Miscellaneous comments*

As we learn from Section 4.2(a), the temperature maximum is not bound to lie exactly in the middle of the gap. Exceptions—among the selected examples—are seen from the foregoing section to be the parallel gap as well as those types of the conic gap and the hyperbolic gap of the second kind which exhibit symmetry with respect to the plane $\vartheta = \pi/2$ (i.e. $z = 0$), provided both walls are at a like temperature (i.e. $\Upsilon = 0$).

The coefficient function g, which is of a purely geometrical character, becomes identical to zero only in the case of the parallel gap and stays nearly zero in some limiting cases, namely—among the selected examples—in the cases of the conic gap and the hyperbolic gaps of either kind in the vicinity of the plane $\vartheta = \pi/2$, and in the case of the elliptic gap of the first kind in the vicinity of the plane $\psi = 0$ (i.e. always in the vicinity of the plane $z = 0$). Note also that g is, on principle, independent of the lateral coordinate ξ^n.

The other coefficient function, f, is not only geometrically conditioned but also contingent upon mechanical entities, which are involved in the quantities α and β. According to (40) α and β are interrelated independently of the lateral position (that is to say, r/h_t does not depend upon ζ^n)—among the selected examples—in the cases of the cylindric and conic gaps only. In these cases the relation between α and β can be varied merely in terms of the cylinder radius r or the cone angle ϑ, respectively. The influence of β increases proportionally to r^2 or $(\sin\vartheta)^2$, respectively. As to the conic pattern, the maximum of this influence belongs to a gap which is symmetric with respect to the plane $\vartheta = \pi/2$, and in this case holds the relation $r/h_t = 1$.

It seems also worth while pointing out the thermorheological import of the quantities α and β. According to the definitions (41) we have

$$\beta/\alpha = (\zeta/\eta)(\partial\omega/\partial\xi^t)^2. \quad (153)$$

By definition, β/α is negative in pseudoplastic fluids. Regarding devices in which the influence of β in proportion to α becomes maximum (such as in the case of a symmetrical conic pattern mentioned just before), one could think of enhancing the gradient $\partial\omega/\partial\xi^t$ so much that the temperature-lowering effect associated with pseudoplasticity comes to outweigh the heating effect of first degree (due to linear viscosity alone), were it not inconsistent with all the experience hitherto gained. At too large rates of shear the ratio of ζ to η remains no longer constant.

In this context we should not omit remarking on the change of the shear rate along the lateral direction in a gap. (This change rests upon the derivative $\partial\dot\gamma/\partial\xi^n$. The formulas for the shear rates $\dot\gamma$ of the various patterns can be looked up in paper [6].) The rate of shear does not vary in the conic and cylindric gaps, but increases with the radial distance in all of the other instances considered in Section 5.2—the sole exception being the hyperbolic gap of the first kind, which behaves just in the opposite sense. An immediate influence upon the temperature distribution cannot be recorded, though.

In view of the theory developed above one may contrive some novel measuring apparatus which are suitable to provide data relating to nonlinear viscosity. The geometrically more complicated patterns would be more informative than those generated by straight profiles of gap. Of course, the latter are easier to manufacture and manipulate, which is the reason of why they are given preference in the conventional viscometry.

6. DETAILS OF SOME APPLICATIONAL EXAMPLES

6.1. *General remarks*

It would be inappropriate in the present paper to elaborate on all of the nine patterns considered in Section 2—with the exception of two rather uncomplicated, but nonetheless interesting, representatives: the parallel-plate and plate-and-cone assemblies, and these only under the boundary conditions of ambidirectional heat transfer according to case (a) as defined in Section 4. The parallel-plate assembly is instructive by virtue of its constructional simplicity, though it may pass as trivial from the standpoint of the present theory insofar as the gap walls are not curved at all. The plate-and-cone assembly has also received some attention in the literature.† In addition, these examples afford partial

† The approach put forward by Bird and Turian [26] was restricted to linearly viscous fluids. These authors used a so-called semidirect variational method. As they supposed the absence of any heat transfer at the gap edge, the underlying boundary conditions differ from those imposed by us. Hence an agreement of the results is demonstrable only in the approximation valid for a region not too far away from the axis of rotation; cf., especially, our equation (164) in what follows.

comparison between our results and those given in the literature.† For the time being, these simple patterns enjoy great favor in applications. Yet this state of affairs ought not to mislead to the belief that other patterns, though more difficult to realize, must remain of minor practical importance for ever.

6.2. *Parallel-plate assembly*

This assembly embodies a parallel gap. In a way, it resembles a slit‡ but, as a body of revolution, exhibits a variable rate of shear which increases proportionally to the radial distance r. This feature perhaps justifies a particular treatment at this place.

Moreover, this example stands out for the property (now to be proved) that the approximate formulas set up in Section 5.2 coincide with the mathematically exact solution. According to (93), f is independent of z or s (since $s = z - z_0$), so that $f' = 0$. Further, we have $g = 0$ and $g' = 0$. Hence we obtain from (53), (54) and (55) $\kappa = 1$, $K = f_0 = f$, $L = 0$ and, therefore, from (52)

$$\theta = \left(\Upsilon + f\frac{S-s}{2}\right)s, \tag{154}$$

from (61)

$$s_{\theta\,\max} = \frac{S}{2} + \frac{\Upsilon}{f} \tag{155}$$

and from (65) in conjunction with (155)

$$\theta_{\max} = \frac{f}{8}S^2 + \frac{\Upsilon}{2}\left(S + \frac{\Upsilon}{f}\right). \tag{156}$$

Indeed, the same formulas emerge from the mathematical exact solution (96).

It follows from (110), or immediately from (155), that the critical values of the temperature difference between the gap walls for the disappearance of the temperature maximum are exactly determined by

$$-\Upsilon_0 = \Upsilon_S = fS/2. \tag{157}$$

The inference that these values differ merely by their signs does no more than reflect the symmetry with respect to the transverse dimension. If, however, one keeps $|\Upsilon|$ fixed and adopts (93), the disappearance of the temperature maximum is confined inside a circular cylinder of radius r_{crit} given by

$$(r_{\text{crit}})^2 = \frac{\alpha}{2\beta}\left[\left(1 + 8\alpha^{-2}\beta\frac{|\Upsilon|}{S}\right)^{1/2} - 1\right] \tag{158}$$

or, if $\beta = 0$ (i.e. in case of linear viscosity),

$$(r_{\text{crit}})^2 = 2|\Upsilon|/\alpha S. \tag{159}$$

Since the second-degree equation (154) is exact, we need not bargain for any points of inflection.

6.3. *Plate-and-cone assembly*

This assembly owes its unsymmetrical configuration to the ease with which it can be built and employed. As regards the temperature distribution, it brings forth departures from that of the symmetrical conic gap only in higher orders if the gap is infinitesimally narrow.

For the example under consideration we have $\vartheta_0 = \pi/2$ and $s = \vartheta - (\pi/2)$. From (101) and (102) we then get

$$f_0 = (\alpha + \beta)\rho^2 \tag{160}$$

and $g_0 = 0$. After forming the derivatives of f and g, we find $f_0' = 0$ and $g_0' = -1$. Hence from (53) through (55) it follows that $\kappa = 1$, $K = f_0$, $L = -\Upsilon$. Thus the approximate formula (52) yields

$$\theta = \left\{f_0\frac{S-s}{2} + [1 - \tfrac{1}{6}(S^2 - s^2)]\Upsilon\right\}s, \tag{161}$$

which holds also in cases not specified by (160) but characterized by the remainder of the above assumptions.

On the other hand, stipulating (160) and inserting the pertinent boundary conditions, we obtain from (104) the mathematically exact expression

$$\begin{aligned}\theta = \Bigg\{\Upsilon S + \rho^2\Bigg[\left(\frac{\alpha}{6} + \frac{7\beta}{30}\right)(\sin S)^2 - \frac{\beta}{20}(\sin S)^4 \\ - 2\left(\frac{\alpha}{3} + \frac{4\beta}{15}\right)\ln\cos S\Bigg]\Bigg\}\frac{\operatorname{artanh}\sin s}{\operatorname{artanh}\sin S} \\ - \rho^2\Bigg[\left(\frac{\alpha}{6} + \frac{7\beta}{30}\right)(\sin s)^2 - \frac{\beta}{20}(\sin s)^4 \\ - 2\left(\frac{\alpha}{3} + \frac{4\beta}{15}\right)\ln\cos s\Bigg]\end{aligned} \tag{162}$$

and, by expanding (162) in terms of powers of s and S up to the fourth degree, the approximate expression

$$\begin{aligned}\theta = \Bigg\{\frac{1}{2}\rho^2\Bigg[(\alpha+\beta)(S-s) - \left(\frac{\alpha}{6} + \frac{\beta}{3}\right)S^3 \\ + \frac{\alpha+\beta}{6}Ss^2 + \frac{\beta}{6}s^3\Bigg] \\ + \Upsilon\left[1 - \frac{1}{6}(S^2 - s^2) - \frac{1}{72}(S^4 + 2S^2s^2 - 3s^4)\right]\Bigg\}s.\end{aligned} \tag{163}$$

Upon dropping all terms of the fourth degree, equation (163) correctly reduces to (161) when (160) is taken into account.

Now we come to inquire into the characteristics of the temperature profile. If we put up with the second-degree approximation, which implies (63) through (65) and (70), we arrive again at the formulas (154) through (157) established for the parallel gap—apart from the distinction between f_0 and f.

† Another instance already treated in the literature is the *coaxial-cylinder assembly*, which furnishes what, in the domain of viscometry, is customarily understood by Couette flows in a restricted sense. But the pertinent formulas put forward by Weltmann and Kuhns [27] seem to be unfortunately in fundamental disagreement from ours.

‡ The *slit*—defined as a pair of parallel walls, accidentally movable against, but maintaining their distance from, each other—can be conceived as the limiting case $r_0 \to \infty$ of a cylindric gap whose walls have lost any curvature. The temperature distribution in a slit came to be treated even in a textbook (viz. [28], esp. pp. 276–279). Calculations based on the slit model for lubricating gaps (cf., e.g. [20], esp. pp. 202–206) have rather vastly proliferated in the engineering literature.

Invoking the next higher degree of approximation stage, we get from (60)

$$s_{\theta\,\max} = \frac{f_0}{\Upsilon} - \left[\left(\frac{f_0}{\Upsilon}\right)^2 - 2 - \frac{f_0}{\Upsilon}S + \frac{S^2}{3}\right]^{1/2}. \quad (164)$$

By inspection of (155) and (47) we infer that $|\Upsilon/f_0|$ must not exceed the order of magnitude of $S/2$ if there is a temperature maximum in the gap. A power series expansion of (164) on this principle of order then yields the approximate expression

$$s_{\theta\,\max} = \frac{S}{2} + \frac{\Upsilon}{f_0}\left[1 + \frac{1}{2}\left(\frac{\Upsilon}{f_0}\right)^2 + \frac{S\Upsilon}{2f_0} + \frac{S^2}{12}\right], \quad (165)$$

which contains in the brackets a correction factor destined for the second-degree approximation corresponding to (155). By the way, the sign given to the square root expression in (164) is thus confirmed.

Instead of (157) we obtain from (164) the unlike values

$$\Upsilon_0 = -f_0 S/(2 - \tfrac{1}{3}S^2), \quad \Upsilon_S = f_0 S/(2 + \tfrac{2}{3}S^2). \quad (166)$$

Upon adopting (160), the disappearance of the temperature maximum for a fixed—either negative or positive—value of Υ turns out to be confined to a region inside a sphere of radius $\rho_{\mathrm{crit}\,0}$ if $\Upsilon < 0$, or $\rho_{\mathrm{crit}\,S}$ if $\Upsilon > 0$, given by

$$(\rho_{\mathrm{crit}\,0})^2 = -\frac{(2 - \tfrac{1}{3}S^2)\Upsilon}{(\alpha+\beta)S}, \quad (\rho_{\mathrm{crit}\,S})^2 = \frac{(2 + \tfrac{2}{3}S^2)\Upsilon}{(\alpha+\beta)S}. \quad (167)$$

Proceeding at the same approximation stage, we get from (62)

$$s_{\theta\,\mathrm{infl}} = (f_0/\Upsilon) - 1. \quad (168)$$

By virtue of (47), the transverse temperature distribution exhibits points of inflection under the criterion $\Upsilon_0 \leqslant \Upsilon \leqslant \Upsilon_S$ with the critical values Υ_0 for $s_{\theta\,\mathrm{infl}} = 0$ and Υ_S for $s_{\theta\,\mathrm{infl}} = S$ given by

$$\Upsilon_0 = f_0, \qquad \Upsilon_S = f_0/(1+S). \quad (169)$$

If, on the other hand, Υ is kept fixed and if f_0 is assigned by (160), points of inflection exist inside the spherical zone

$$\left(\frac{\Upsilon}{\alpha+\beta}\right)^{1/2} \leqslant \rho \leqslant \left[\frac{\Upsilon}{\alpha+\beta}(1+S)\right]^{1/2}. \quad (170)$$

Although, setting out from the exact solution (162), one is able to show that the accuracy of the approximate formulas (168) through (170) will be modest, we renounce presenting further computations within the framework of this paper.

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CALCUL DES CHAMPS DE TEMPERATURE STATIONNAIRES DANS LES ECOULEMENTS DE COUETTE GENERALISES DE FLUIDES SIMPLES

Résumé—Une équation différentielle fondamentale du bilan local de flux d'énergie dans des fluides homogènes simples (au sens de Noll) est étendue au cas de mouvements cisaillés quasi-simples spatialement et matériellement uniformes le long de lignes de courants circulaires (ou parallèles). A l'aide de quelques théorêmes connus de la théorie non-linéaire des milieux continus, on montre que les mouvements supposés (ici brièvement désignés par écoulement de Couette généralisés) sont dynamiquement à peu près possibles s'il sont limités à des passages très étroits, et que la réponse physique du seul premier degré dans les tenseurs cinématiques, représentée par la fonction viscosité, recouvre entièrement le phénomène du flux de travail.

L'équation aux dérivées partielles établie sous forme générale est ensuite réduite à une équation différentielle ordinaire en imposant des conditions aux limites qui permettent de négliger toutes les dérivées de la température autres que celles par rapport à la direction transversale (c'est à dire perpendiculaire aux parois de l'espace étroit). Dans la direction longitudinale (c'est à dire le long des lignes de courant), la température ne varie pas d'après l'hypothèse de symétrie de révolution. Dans la direction latérale suivant laquelle la largeur du passage peut varier lentement, tout effet de bord est négligé. Suivant la direction transversale, deux cas sont considérés: (a) chacune des deux parois du passage sont maintenues à des températures constantes éventuellement différentes, (b) une paroi est a température fixée mais l'autre paroi est calorifugée si bien que sa température dépend de la production de chaleur. Des formules approchées, basées, sur ces hypothèses et sur un développement de Taylor suivant la coordonnée transversale, sont établies dans le cas de champs de température dans des espaces de révolution de section droite quelconque.

En introduisant des systèmes de coordonnées orthogonales spécifiques adaptés à des configurations variées de passages étroits de révolution (à savoir: cylindriques, parallèles, coniques, sphériques, paraboliques, hyperboliques et elliptiques), la voie est ouverte pour obtenir des solutions définies et mathématiquement exactes. On tient compte de la dépendance de la viscosité sur le taux de cisaillement dans une première approximation basée sur un développement en série entière en fonction de l'invariant cinématique. Un ensemble de formules de base pour diverses configurations de passages sont données essentiellement à titre de référence. Dans la conclusion, quelques assemblages fréquemment utilisés (plaques parallèles, plaque et cone) sont étudiés en détail.

BERECHNUNG STATIONÄRER TEMPERATURFELDER IN VERALLGEMEINERTEN COUETTESTRÖMUNGEN EINFACHER FLÜSSIGKEITEN

Zusammenfassung—Eine grundlegende Differentialgleichung der lokalen Bilanz des Energieflusses in homogenen einfachen Flüssigkeiten (im NOLL-schen Sinne) wird für räumlich und materiell stationäre, quasi-einfache Scherbewegungen längs kreisförmiger (oder auch paralleler) Stromlinien neu entwickelt. Unter Benutzung einiger aus der nichtlinearen Kontinuumstheorie bekannter Lehrsätze wird gezeigt, daß die vorausgesetzten—hier kurz als verallgemeinerte Couetteströmungen bezeichneten—Bewegungen dynamisch fast möglich sind, wenn sie auf sehr enge Spalte beschränkt werden und daß das Materialverhalten ersten Grades in den kinematischen Tensoren, wie es durch die Viskositätsfunktion wiedergegeben wird, Erscheinungen des Arbeitsflusses erschöpfend erfaßt.

Die in allgemeiner Form aufgestellte partielle Differentialgleichung wird dann auf eine gewöhnliche zurückgeführt, indem Randbedingungen angenommen werden, die es gestatten, alle Ableitungen der Temperatur außer denen nach der transversalen Richtung (d. h. senkrecht zu den Wänden des engen Spaltes) zu vernachlässigen. In der longitudinalen Richtung (d. h. längs der Stromlinie), ändert sich die Temperatur dank der vorausgesetzten Rotationssymmetrie nicht. In der lateralen Richtung, in der sich die Spaltweite langsam ändern kann, bleiben etwaige Randeffekte außer Betracht. In transversaler Richtung werden zwei Fälle berücksichtigt: (a) beide Wände des Spaltes werden auf konstanten, möglicherweise unterschiedlichen Temperaturen gehalen; (b) eine Wand hat eine feste Temperatur, die andere aber ist gegen Wärmeleitung isoliert, so daß ihre Temperatur von der Wärmeerzeugung abhängt. Aufgrund dieser Annahmen und einer Taylor-Entwicklung in der transversalen Koordinate werden Näherungsformeln für Temperaturfelder in Umdrehungsspalten beliebiger Querschnittsform errechnet.

Durch Festlegen besonderer rechtwinkliger Koordinatensysteme, die zu verschiedenen—nämlich zylindrischen, parallelen, konischen, sphärischen, parabolischen, hyperbolischen und elliptischen—Formen enger Umdrehungsspalte passen, wird der Weg zum Angeben bestimmter und mathematisch exakter Lösungen bereitet. Die Abhängigkeit der Viskosität von der Schergeschwindigkeit wird in einer ersten Näherung berücksichtigt, die auf einer Potenzreihenentwicklung nach der kinematischen Invariante beruht. Eine Sammlung von Grundformeln für die verschiedenartigen Spaltformen wird hauptsächlich zu Nachschlagezwecken gegeben. Am Schluß werden einige oft benutzte Anordnungen (parallele Platten, Platte und Kegel) im einzelnen untersucht.

РАСЧЕТ СТАЦИОНАРНЫХ ТЕМПЕРАТУРНЫХ ПОЛЕЙ ОБОБЩЕННЫХ ТЕЧЕНИЙ КУЭТТА ДЛЯ ПРОСТЫХ ЖИДКОСТЕЙ

Аннотация — Выведено основное дифференциальное уравнение локального баланса потока энергии в однородных простых жидкостях (в смысле Нолля) для пространственных стационарных квазипростых сдвиговых течений вдоль круговых (а также параллельных) линий тока.

С помощью некоторых теорем, известных из нелинейной теории сплошных сред, показано, что предполагаемые течения (здесь кратко называемые обобщенными течениями Куэтта) динамически почти возможны, если они ограничены очень узкими зазорами. Кроме того, материальная характеристика первой степени в кинематических тензорах, представленная функцией вязкости, полностью описывает энергетический поток. Дифференциальное уравнение в частных производных, полученное в общем виде, затем сводится к обыкновенному дифференциальному уравнению для граничных условий, позволяющих пренебречь всеми производными от температуры, кроме тех, которые взяты в поперечном направлении (т. е. перпендикулярно стенкам узкого зазора). В продольном направлении (т. е. вдоль линий тока) температура не изменяется вследствие предполагаемой симметрии вращения. Кроме того, когда ширина зазора может слегка изменяться, любые краевые эффекты не учитываются. Что касается поперечного направления, рассматриваются два случая, (А) температуры обеих стенок, возможно различные, поддерживаются постоянными; (Б) температура одной стенки фиксирована, а другая стенка теплоизолирована, так, что её температура зависит от количества выделяемого тепла.

Исходя из этих предположений и разложения в ряд Тейлора по поперечным координатам, получены приближенные формулы для температурных полей в зазорах, образующихся при вращении тел любого сечения. Использование определенных ортогональных систем координат, соответствующих различным, а именно цилиндрическим, параллельным, коническим, сферическим, параболическим и эллиптическим формам узких зазоров вращения позволяет получить определенные и математически точные решения. Зависимость вязкости от скорости сдвига учитывается в первом приближении на основе степенного разложения в ряд, исходя из кинематического инварианта. Целый ряд основных формул для различных форм зазоров приводится, в основном, для справки. В заключение подробно рассматриваются некоторые часто используемые комбинации (параллельные пластины, пластина и конус).